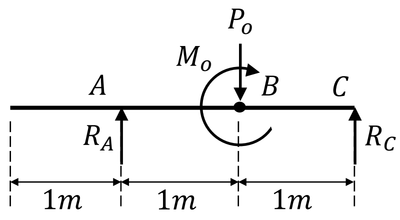
2017-2018 MM2MS3 Exam Solutions

1.



Vertical equilibrium of the beam:

$$R_A + R_C = P_o \qquad (1)$$

[2 marks]

Taking moments about position C:

$$P_o \times 1 = R_A \times 2 + M_o$$

$$\therefore R_A = \frac{P_o - M_o}{2}$$

[2 marks]

Substituting values of P_o and M_o gives:

$$R_A = 1{,}500 \text{ N}$$

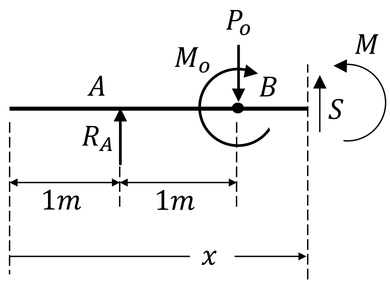
[1 mark]

Rearranging (1) for R_C and substituting values for R_A and P_o gives:

$$R_C = 3{,}500 \text{ N}$$

[1 mark]

Taking the origin at the left-hand end of the beam, sectioning after the last discontinuity and drawing a free body diagram of the left-hand side of the section:



[3 marks]

Taking moments about the section position:

$$R_A\langle x-1\rangle + M_o\langle x-2\rangle^0 = M + P_o\langle x-2\rangle$$

$$\therefore M = R_A\langle x-1\rangle + M_o\langle x-2\rangle^0 - P_o\langle x-2\rangle$$

[2 marks]

Substituting this into the main deflections of beams equation ($EI\frac{d^2y}{dx^2} = M$):

$$EI\frac{d^2y}{dx^2} = R_A\langle x-1\rangle + M_o\langle x-2\rangle^0 - P_o\langle x-2\rangle$$

[2 marks]

Integrating with respect to x:

$$EI\frac{dy}{dx} = \frac{R_A\langle x-1\rangle^2}{2} + M_o\langle x-2\rangle - \frac{P_o\langle x-2\rangle^2}{2} + A \qquad (2)$$

[2 marks]

Integrating with respect to x again:

$$EIy = \frac{R_A\langle x-1\rangle^3}{6} + \frac{M_o\langle x-2\rangle^2}{2} - \frac{P_o\langle x-2\rangle^3}{6} + Ax + B \qquad (3)$$

[2 marks]

Boundary conditions:

(BC1) At $x = 1$, $y = 0$, therefore from (3):

$$A + B = 0 \qquad (4)$$

[2 marks]

(BC2) At $x = 3$, $y = 0$, therefore from (3):

$$0 = \frac{R_A \times 2^3}{6} + \frac{M_o \times 1^2}{2} - \frac{P_o \times 1^3}{6} + 3A + B$$

$$\therefore A = \frac{P_o - 3M_o - 8R_A}{12}$$

Substituting values of P_o, R_A and M_o into this gives:

$$A = -1{,}083.33$$

Therefore, from (4):

$$B = 1{,}083.33$$

[2 marks]

From (2), at $x = 2$ (point B):

$$\frac{dy}{dx} = \frac{1}{EI}\left(\frac{R_A\langle x-1\rangle^2}{2} + M_o\langle x-2\rangle - \frac{P\langle x-2\rangle^2}{2} + A\right)$$

Substituting values of E, I, R_A, M_o, P_o and A into this gives:

$$\boldsymbol{\frac{dy}{dx} = -0.000833\ \mathbf{rads}}$$

[2 marks]

From (3), at $x = 2$ (point B):

$$y = \frac{1}{EI}\left(\frac{R_A\langle x-1\rangle^3}{6} + \frac{M_o\langle x-2\rangle^2}{2} - \frac{P_o\langle x-2\rangle^3}{6} + Ax + B\right)$$

Substituting values of E, I, R_A, M_o, P_o, A and B into this gives:

$$\boldsymbol{y = 0.001042\ \mathbf{m} = 1.042\ \mathbf{mm}}$$

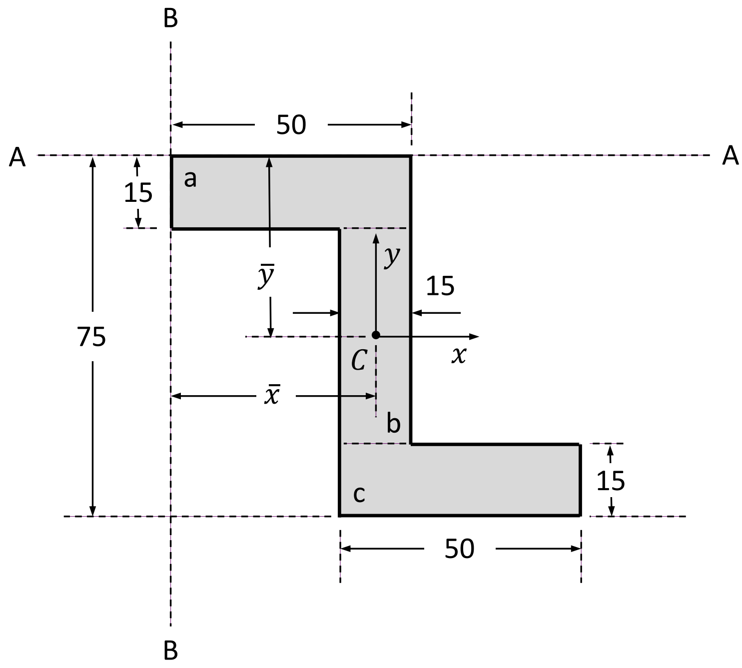
(i.e. upward deflection)

[2 marks]

2.

(a)

Position of Centroid, C



Total area,

$$A = (50 \times 15)_a + (15 \times 45)_b + (50 \times 15)_c = 2{,}175 \text{ mm}^2$$

Taking moments about AA:

$$\bar{y} = \frac{(50 \times 15 \times 7.5)_a + (15 \times 45 \times 37.5)_b + (50 \times 15 \times 67.5)_c}{2{,}175}$$

$$\therefore \bar{\boldsymbol{y}} = \mathbf{37.5\ mm}$$

[3 marks]

Similarly, taking moments about BB:

$$\bar{x} = \frac{(15 \times 50 \times 25)_a + (45 \times 15 \times 42.5)_b + (15 \times 50 \times 60)_c}{2{,}175}$$

$$\therefore \bar{\boldsymbol{x}} = \mathbf{42.5\ mm}$$

[3 marks]

(b)

Principal 2nd Moments of Area

Using the Parallel Axis Theorem,

$$I_{x\prime} = (I_x + Ab^2)_a + (I_x + Ab^2)_b + (I_x + Ab^2)_c$$

$$= \left(\frac{50 \times 15^3}{12} + 50 \times 15 \times 30^2\right) + \left(\frac{15 \times 45^3}{12} + 15 \times 45 \times 0^2\right) + \left(\frac{50 \times 15^3}{12} + 50 \times 15 \times -30^2\right)$$

$$= 1{,}492{,}031.25 \text{ mm}^4$$

[2 marks]

and,

$$I_{y\prime} = \left(I_y + Aa^2\right)_a + \left(I_y + Aa^2\right)_b + \left(I_y + Aa^2\right)_c$$

$$= \left(\frac{15 \times 50^3}{12} + 15 \times 50 \times -17.5^2\right) + \left(\frac{45 \times 15^3}{12} + 45 \times 15 \times 0^2\right) + \left(\frac{15 \times 50^3}{12} + 15 \times 50 \times 17.5^2\right)$$

$$= 784{,}531.25 \text{ mm}^4$$

[2 marks]

Also,

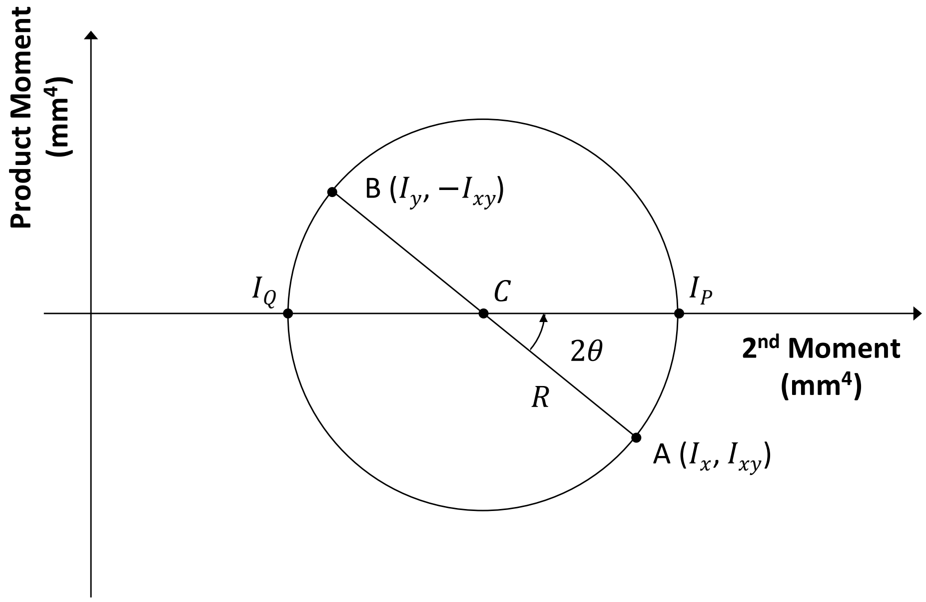
$$I_{x\prime y\prime} = \left(I_{xy} + Aab\right)_a + \left(I_{xy} + Aab\right)_b + \left(I_{xy} + Aab\right)_c$$

$$= (0 + 50 \times 15 \times -17.5 \times 30) + (0 + 15 \times 45 \times 0 \times 0) + (0 + 50 \times 15 \times 17.5 \times -30)$$

$$= -787{,}500 \text{ mm}^4$$

[2 marks]

Mohr's Circle:



$$Centre, C = \frac{I_{x\prime} + I_{y\prime}}{2} = \frac{1{,}492{,}031.25 + 784{,}531.25}{2} = 1{,}138{,}281.25 \text{ mm}^4$$

$$Radius, R = \sqrt{\left(\frac{I_{x'} - I_{y'}}{2}\right)^2 + {I_{x'y'}}^2} = \sqrt{\left(\frac{1{,}492{,}031.25 - 784{,}531.25}{2}\right)^2 + (-787{,}500)^2}$$

$$= 863{,}304.88 \text{ mm}^4$$

[3 marks]

Therefore, the Principal 2nd Moments of Area are:

$$I_P = C + R = 1{,}138{,}281.25 + 863{,}304.88$$

$$\therefore \boldsymbol{I_P = 2{,}001{,}586.13\ \text{mm}^4}$$

[2 marks]

and,

$$I_Q = C - R = 1{,}138{,}281.25 - 863{,}304.88$$

$$\therefore \boldsymbol{I_Q = 274{,}976.37\ \text{mm}^4}$$

[2 marks]

(c)

Orientation of the Principal Axes with respect to the x-y co-ordinate system

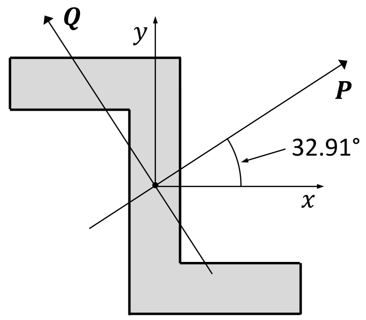
From the Mohr's circle above:

$$sin2\theta = \frac{I_{xy}}{R} = \frac{-787{,}500}{863{,}304.88}$$

$$\therefore \theta = -32.91°$$

[3 marks]

Therefore, **the Principal Axes are at -32.91° (anti-clockwise) from the x-y axes**, as shown on the diagram below.



[3 marks]

3.

(a)

$$\sigma_r = A - \frac{B}{r^2} - \frac{3+\nu}{8}\rho\omega^2 r^2$$

$$\sigma_\theta = A + \frac{B}{r^2} - \frac{1+3\nu}{8}\rho\omega^2 r^2$$

[2 marks]

at $r = 0.05$ (ID), $\sigma_r = 0$ therefore:

$$0 = A - \frac{B}{0.05^2} - \frac{3+0.3}{8}7900 \times 523.6^2 \times 0.05^2$$

$$\therefore 0 = A - 400B - 223.4 \times 10^4 \qquad (1)$$

or,

$$A - 400B - 8.15\omega^2$$

[3 marks]

at $r = 0.5$ (OD), $\sigma_r = 0$ therefore:

$$0 = A - \frac{B}{0.5^2} - \frac{3+0.3}{8}7900 \times 523.6^2 \times 0.5^2$$

$$\therefore 0 = A - 4B - 223.4 \times 10^6 \qquad (2)$$

or,

$$A - 4B - 814.6\omega^2$$

[3 marks]

substituting (2) from (1):

$$0 = -396B + 221.16 \times 10^6$$

[1 mark]

therefore:

$$B = 558500 \text{ (or } 2.04\omega^2)$$

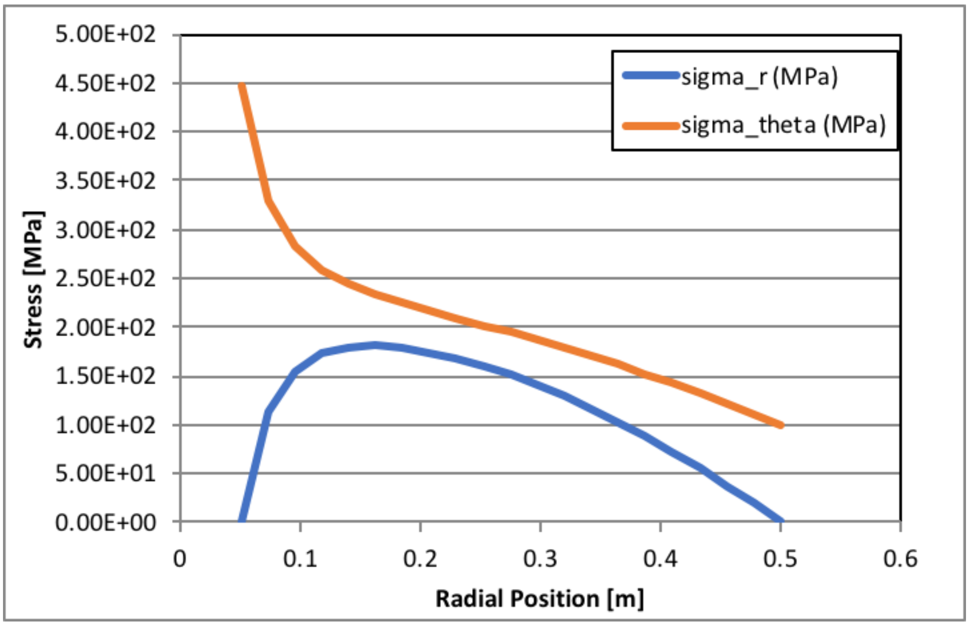
and from (1),

$$A = 225.6 \times 10^6 \text{ (or } 822.7\omega^2)$$

[2 marks]

r	sigma_r (MPa)	sigma_theta (MPa)
5.00E-02	0.00E+00	4.48E+02
0.15	**1.81E+02**	**2.39E+02**
0.3	**1.39E+02**	**1.85E+02**
0.5	**0.00E+00**	**9.92E+01**

[2 marks]



[2 marks]

(b)

Max sigma theta is at bore ($r = 0.05$), therefore:

$$\sigma_{\theta_{max}} = 822.7\omega^2 + \frac{2.04\omega^2}{r^2} - \frac{1 + 3 \times 0.3}{8} 7900\omega^2 0.05^2$$

$$\sigma_{\theta_{max}} = \omega^2 \left(822.7 + \frac{2.04}{0.05^2} - 1185 \times 0.05^2\right) = 1635.7\omega^2$$

[4 marks]

Hoop stress is limited to 240 MPa, therefore at the ID,

$$240 \times 10^6 = 1635.7\omega^2$$

$$\omega^2 = 146723$$

$$\boldsymbol{\omega = 383\ \mathrm{rad/s}}$$

$$383\ \mathbf{rad/s} = \frac{\omega}{2\pi} \times 60 = 3657\ \mathbf{rpm}$$

[6 marks]

4.

(a)

The shear stress distribution in the web is given by:

$$\tau_{flange} = \frac{SA\bar{y}}{Iz} = \frac{S(at)d}{2It} = \frac{Sda}{2I}$$

where,

$$I = \frac{bd^3 - (d-2t)^3(b-t)}{12}$$

therefore,

$$\tau_{flange} = \frac{6Sda}{bd^3 - (d-2t)^3(b-t)}$$

[4 marks]

at free surface,

$$\boldsymbol{\tau_{flange0} = 0}$$

[2 marks]

$$\therefore \tau_{flange28} = \frac{6 \times 5000 \times 40 \times 28}{30 \times 40^3 - (40 - 2 \times 2)^3(30-2)} = \mathbf{54.8\ MPa}$$

[2 marks]

(b)

$$\tau_{web} = \frac{SA\bar{y}}{Iz} = \frac{S}{It}\left[\frac{btd}{2} + \left(\frac{d}{2} - y\right)t\left(\frac{d}{2} + y\right)\frac{1}{2}\right] = \frac{S}{2I}\left(bd + \left(\frac{d}{2}\right)^2 - y^2\right)$$

$$\therefore \tau_{web} = \frac{6S}{bd^3 - (d-2t)^3(b-t)}\left(bd + \left(\frac{d}{2}\right)^2 - y^2\right)$$

[4 marks]

$$\tau_{web0} = \frac{6 \times 5000}{30 \times 40^3 - (40 - 2 \times 2)^3(30-2)} \times \left((30 \times 40) + \left(\frac{40}{2}\right)^2 - 0^2\right) = \mathbf{78.2\ MPa}$$

[2 marks]

$$\tau_{web18} = \frac{6 \times 5000}{30 \times 40^3 - (40 - 2 \times 2)^3 (30 - 2)} \times \left((30 \times 40) + \left(\frac{40}{2}\right)^2 - 19^2 \right) = \mathbf{62.4\ MPa}$$

[2 marks]

(c)

The force given by the shear stress in the flange (S_1) is;

$$S_1 = \int_0^b \tau t da = \int_0^b \frac{Sda}{2I} t da = \frac{Sdtb^2}{4I}$$

[3 marks]

If we take moments about O in the web (where e is the distance along the N.A. from O):

$$Se = 2S_1 \frac{d}{2}$$

[3 marks]

Therefore:

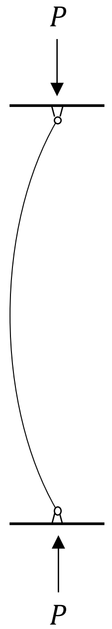
$$e = \frac{S_1 d}{S} = \frac{d^2 t b^2}{4I} = \frac{3 \times (40^2 \times 2 \times 30^2)}{(30 \times 40^3 - (40 - 2 \times 2)^3 (30 - 2))} = \mathbf{14.1mm}$$

[3 marks]

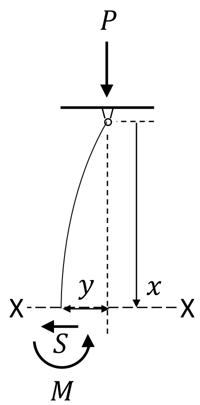
5.

(a)

Below is a diagrammatic representation of a pinned-pinned strut.



Sectioning this beam in order to determine the bending moment:



[1 mark]

Taking moments about the section position, X-X:

$$M = Py \qquad (1)$$

2nd order differential equation for a beam under bending:

$$EI\frac{d^2y}{dx^2} = M$$

Substituting (1) into this:

$$EI\frac{d^2y}{dx^2} = Py$$

[1 mark]

Let $y = A_O e^{\alpha x}$:

$$\therefore EI\alpha^2 + P = 0$$

and:

$$\alpha = \pm\sqrt{\frac{P}{EI}}i$$

[1 mark]

We get:

$$y = Asin\left(\sqrt{\frac{P}{EI}}x\right) + Bcos\left(\sqrt{\frac{P}{EI}}x\right) \qquad (2)$$

where A_O, A and B are constants.

Boundary conditions:

(BC1) At $x = 0$, $y = 0$, therefore from (2):

$$B = 0$$

[1 mark]

(BC2) At $x = L$, $y = 0$, therefore from (2):

$$Asin\left(\sqrt{\frac{P}{EI}}L\right) = 0$$

[1 mark]

Since $A \neq 0$ for non-trivial solution:

$$\sqrt{\frac{P}{EI}}L = n\pi$$

$$\therefore \boldsymbol{P} = \frac{\boldsymbol{n^2\pi^2EI}}{\boldsymbol{L^2}} \qquad (3)$$

where $n = 1, 2, \ldots$

[2 marks]

(b)

$$\sigma = \frac{P}{A} \qquad (4)$$

[2 marks]

Substituting the buckling load (from (3)) into equation (4):

$$\sigma = \frac{n^2\pi^2EI}{L^2A} = \frac{n^2\pi^2EAk^2}{L^2A} = \frac{\pi^2E}{\left(\frac{L}{k}\right)^2}$$

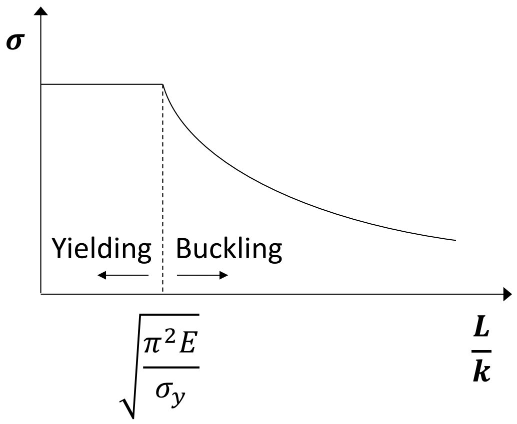
[2 marks]

Therefore, when $\sigma = \sigma_y$:

$$\frac{L}{k} = \sqrt{\frac{\pi^2E}{\sigma_y}}$$

[2 marks]

The following figure therefore displays the transition between yielding and buckling in terms of slenderness ratio.



[4 marks]

(c)

In the fixed-fixed case,

$$P = \frac{4\pi^2 EI}{L^2}$$

Substituting this into equation (3):

$$\sigma = \frac{4\pi^2 EI}{L^2 A} = \frac{4\pi^2 EAk^2}{L^2 A} = \frac{4\pi^2 E}{\left(\frac{L}{k}\right)^2}$$

Therefore, when $\sigma = \sigma_y$:

$$\frac{L}{k} = \sqrt{\frac{4\pi^2 E}{\sigma_y}} = 2\pi\sqrt{\frac{200{,}000}{250}} = 177.72 \qquad (5)$$

where k is calculated by:

$$I = \frac{bd^3}{12} = \frac{t^4}{12} = t^2 k^2$$

where $b = d = t$ and $I = Ak^2 = bdk = t^2k^2$.

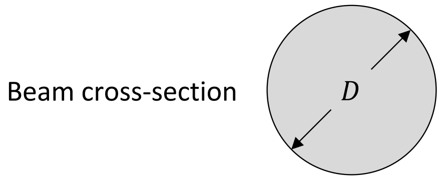
$$\therefore k = 12.99 \text{ mm}$$

Therefore from (5):

$$\boldsymbol{L = 2{,}308.58 \text{ mm} = 2.309 \text{ m}}$$

6.

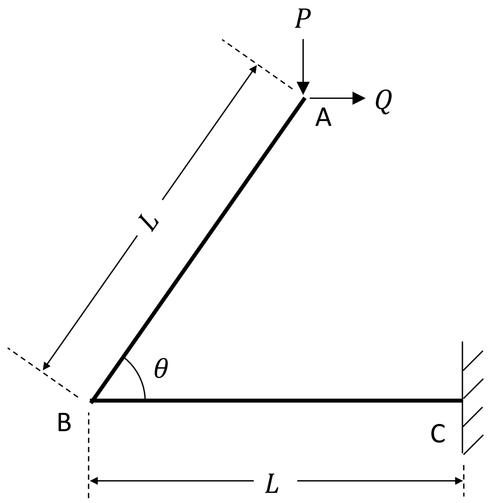
Second Moment of Area, I, calculation:



$$\therefore I = \frac{\pi D^4}{64} = \frac{\pi \times 40^4}{64} = 125{,}663.71\ \mathrm{mm}^4$$

[2 marks]

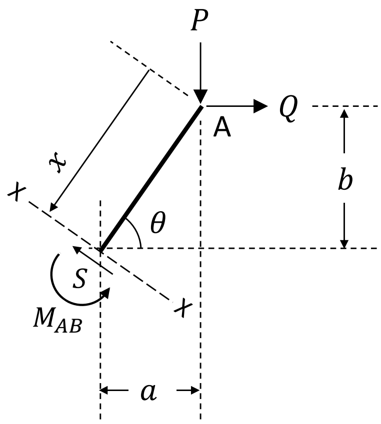
Adding dummy load, Q, and labelling the structure:



[2 marks]

Section AB (*bending only*)

Free Body Diagram:



[2 marks]

Taking moments about X-X:

$$M_{AB} = Pa + Qb = Pxcos\theta + Qxsin\theta$$

[1 mark]

Substituting this into the equation for Strain Energy in a beam under bending gives,

$$U_{AB} = \int \frac{{M_{BC}}^2}{2EI} ds = \int_0^L \frac{(Pxcos\theta + Qxsin\theta)^2}{2EI} dx = \frac{1}{2EI}\int_0^L (P^2x^2cos^2\theta + Q^2x^2sin^2\theta + PQx^2cos\theta sin\theta)dx$$

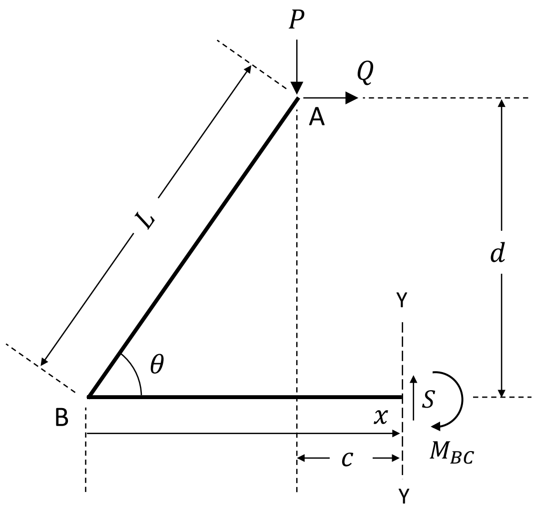
$$= \frac{(P^2cos^2\theta + Q^2sin^2\theta + PQcos\theta sin\theta)}{2EI}\int_0^L x^2dx = \frac{(P^2cos^2\theta + Q^2sin^2\theta + PQcos\theta sin\theta)}{2EI}\left[\frac{x^3}{3}\right]_0^L$$

$$\therefore U_{AB} = \frac{L^3}{6EI}(P^2cos^2\theta + Q^2sin^2\theta + PQcos\theta sin\theta)$$

[2 marks]

Section BC (*bending only*)

Free Body Diagram:



[2 marks]

Taking moments about Y-Y:

$$M_{BC} + Qd = Pc$$

$$\therefore M_{BC} = Pc - Qd = P(x - Lcos\theta) - QLsin\theta$$

[1 mark]

Substituting this into the equation for Strain Energy in a beam under bending gives,

$$\therefore U_{BC} = \int \frac{{M_{BC}}^2}{2EI} ds = \int_0^L \frac{(P(x - Lcos\theta) - QLsin\theta)^2}{2EI} dx$$

$$= \frac{1}{2EI}\int_0^L (P^2x^2 - 2P^2Lxcos\theta + P^2L^2cos^2\theta - 2PQLxsin\theta + Q^2L^2sin^2\theta + 2PQL^2cos\theta sin\theta)dx$$

$$= \frac{1}{2EI}\left[\frac{P^2x^3}{3} - P^2Lx^2cos\theta + P^2L^2xcos^2\theta - PQLx^2sin\theta + Q^2L^2xsin^2\theta + 2PQL^2xcos\theta sin\theta\right]_0^L$$

$$\therefore U_{BC} = \frac{L^3}{2EI}\left(\frac{P^2}{3} - P^2cos\theta + P^2cos^2\theta - PQsin\theta + Q^2sin^2\theta + 2PQcos\theta sin\theta\right)$$

[2 marks]

Total Strain Energy

$$U = U_{AB} + U_{BC}$$

$$\therefore U = \frac{L^3}{6EI}(P^2cos^2\theta + Q^2sin^2\theta + PQcos\theta sin\theta)$$
$$+\frac{L^3}{2EI}\left(\frac{P^2}{3} - P^2cos\theta + P^2cos^2\theta - PQsin\theta + Q^2sin^2\theta + 2PQcos\theta sin\theta\right) \quad (1)$$

[3 marks]

Vertical deflection at position A, u_{v_A}

Differentiating (1) with respect to the applied load, P:

$$u_{v_A} = \frac{\partial U}{\partial P} = \frac{L^3}{6EI}(2Pcos^2\theta + Qcos\theta sin\theta) + \frac{L^3}{2EI}\left(\frac{2P}{3} - 2Pcos\theta + 2Pcos^2\theta - Qsin\theta + 2Qcos\theta sin\theta\right)$$

[2 marks]

Setting dummy load to zero,

$$u_{v_A} = \frac{PL^3cos^2\theta}{3EI} + \frac{PL^3}{EI}\left(\frac{1}{3} - cos\theta + cos^2\theta\right)$$

$$= \frac{PL^3}{EI}\left(\frac{4}{3}cos^2\theta - cos\theta + \frac{1}{3}\right)$$

$$= \frac{16{,}000 \times 750^3}{225{,}000 \times 125{,}663.71}\left(\frac{4}{3}cos^2(55) - cos(55) + \frac{1}{3}\right)$$

$$\therefore \boldsymbol{u_{v_A} = 47.38\ \mathbf{mm}}$$

[2 marks]

Horizontal deflection at position A, u_{h_A}

Differentiating (1) with respect to the dummy load, Q:

$$u_{h_A} = \frac{\partial U}{\partial Q} = \frac{L^3}{6EI}(2Qsin^2\theta + Pcos\theta sin\theta) + \frac{L^3}{2EI}(-Psin\theta + 2Qsin^2\theta + 2Pcos\theta sin\theta)$$

[2 marks]

Setting dummy load to zero,

$$u_{h_A} = \frac{L^3}{6EI}(Pcos\theta sin\theta) + \frac{L^3}{2EI}(2Pcos\theta sin\theta - Psin\theta)$$

$$= \frac{PL^3}{EI}\left(\frac{7cos\theta sin\theta}{6} - \frac{1}{2}sin\theta\right)$$

$$= \frac{16{,}000 \times 750^3}{225{,}000 \times 125{,}663.71}\left(\frac{7cos(55)sin(55)}{6} - \frac{1}{2}sin(55)\right)$$

$$\therefore \boldsymbol{u_{h_A} = 33.09\ \mathrm{mm}}$$

[2 marks]